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ORIGINAL PAPER

Physical invariant strain energy function for passive myocardium

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Abstract Principal axis formulations are regularly used in isotropic elasticity, but they are not often used in dealing with anisotropic problems. In this paper, based on a principal axis technique, we develop a physical invariant orthotropic constitutive equation for incompressible solids, where it contains only a one variable (general) function. The corresponding strain energy function depends on six invariants that have immediate physical interpretation. These invariants are useful in facilitating an experiment to obtain a specific constitutive equation for a particular type of materials. The explicit appearance of the classical ground-state constants in the constitutive equation simplifies the calculation for their admissible values. A specific constitutive model is proposed for passive myocardium, and the model fits reasonably well with existing simple shear and biaxial experimental data. It is also able to predict a set of data from a simple shear experiment.

Keywords Strain energy · Myocardium · Invariants with immediate physical meaning · Orthotropic · Principal axes · Incompressible

1 Introduction

Excellent and comprehensive analyses on mechanical properties of passive myocardium have recently been carried out by Holzapfel and Ogden (2009a). They also excellently discussed several constitutive models that appeared in the literature, and readers are referred to their paper for detailed information. In isotropic elasticity, phenomenological strain energy functions with principal stretches have certain attractive features from both the mathematical and physical viewpoints (Ogden 1972; Valanis and Landel 1967). These forms of strain energy have been widely and successfully used in predicting elastic deformations (Ogden 1972; Shariff 2000; Marckmann and Verron 2006). The Valanis and Landel (1967) isotropic strain energy function has not only been successful in modelling various types of isotropic solid (Shariff 2000), but is also simple in form in the sense that it uses only a general single variable function. In addition to the above attractive features, we note that it is easier to analyse the stress-softening behaviour of anisotropic soft tissues using principal stretches (Dorfmann et al. 2007).

Inspired by the principal stretch successes and the simple form of Valanis and Landel (1967), in this communication, we construct a strain energy function that contains only a general single variable function. We propose a constitutive equation based on the recent principal axis formulation of Shariff (2011) for orthotropic solids. The proposed strain energy function for the constitutive equation depends on six simple invariants that have immediate physical interpretation. Two of the invariants are the principal extension ratios λ_i $(i = 1, 2)$, and the other four are $1 \geq \zeta_i = (\boldsymbol{a} \bullet \boldsymbol{e}_i)^2 \geq 0$ and $1 \geq \xi_i = (\boldsymbol{b} \bullet \boldsymbol{e}_i)^2 \geq 0$ $(i = 1, 2)$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are any two principal directions of the right stretch tensor $\boldsymbol{U}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are the preferred orthogonal directions of the orthotropic solid. The physical meaning of λ_i is obvious, and it is clear that ζ_i and ξ_i are the square of the cosine of the angle between the principal direction $\boldsymbol{e}_i$ and the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. A strain energy formulation using non-immediate-physical interpretation invariants is, in general, not experimentally friendly. For example, an isochoric uniaxial stretch in one of the preferred directions will perturb all the classical invariants

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given in Eq. (2.5), which is not ideal in obtaining a specific form of strain energy function if the specific form is determined by doing tests that vary one invariant and hold the rest of the invariants constant. However, the immediate-physical-interpretation invariants used here are experimentally friendly as described in Shariff (2011). In Sect. 2, a symmetrical general strain energy function expressed in terms of the immediate-physical-interpretation invariants is introduced. Using these invariants, the incompressible ground-state conditions for orthotropic materials are easily derived in Sect. 2. We note that the ground-state conditions are rarely derived for other types of invariants that appeared in the literature. The theoretical results for biaxial and simple shear deformations given in Sect. 3 are used in Sect. 6, where we compare our theory with experimental data. When a nonlinear incompressible orthotropic strain energy function is specialized to classical (infinitesimal) elasticity, it should contain six independent classical ground-state constants (Spencer 1984) to fully characterize an arbitrary material in infinitesimal strain deformations. In Sect. 4, a constitutive model is proposed; It contains only a general single variable function s, and the six independent classical ground-state constants (Spencer 1984) appear explicitly. A specific form of s is proposed for passive myocardium soft tissue. One advantage of having the ground-state constants appear explicitly in the model is that we could easily put restrictions on their values (for physically reasonable responses), and this is done in Sect. 5, where a restriction on the function s is also given. Finally, in Sect. 6, our specific constitutive equation is curve fitted to Dokos et al. (2002) simple shear and Yin et al. (1987) biaxial data. We only curve fit our model to five of the six sets of simple shear data and successfully predict the sixth set of data.

2 Strain energy function with physical invariants

We first recall some essential kinematics of finite deformation of an orthotropic elastic solid. Consider a body occupying the region B_0 in some reference configuration. Let $\boldsymbol{F}$ be the deformation tensor and $\boldsymbol{X}$ a position vector of a point in B_0. Under this deformation, the point moves to a new position $\boldsymbol{x}(\boldsymbol{X}) \in B$, where B is the current configuration of the deformed body. The principal stretch λ_i $(i = 1, 2, 3)$ is given by

$$\lambda_i = \boldsymbol{e}_i \bullet \boldsymbol{U}\boldsymbol{e}_i, \tag{2.1}$$

where $\boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}$. In this communication, all subscripts i and j take the values 1, 2 and 3, unless stated otherwise.

In view of the work of Holzapfel and Ogden (2009a), a passive myocardium tissue can be treated as an incompressible orthotropic material with the preferred orthogonal directions $\boldsymbol{a}$ and $\boldsymbol{b}$. For an incompressible material, we have the constraint $\lambda_1\lambda_2\lambda_3 = 1$. As mentioned in Sect. 1, recently, Shariff (2011) developed a strain energy function W_e for an incompressible orthotropic material, where its invariants have immediate physical interpretation. It has the form

$$\begin{aligned} W_e &= W(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) \\ &= \tilde{W}\left(\lambda_1, \lambda_2, \lambda_3 = \frac{1}{\lambda_1\lambda_2}, \zeta_1, \zeta_2, \xi_1, \xi_2\right) \end{aligned} \tag{2.2}$$

In this paper, we let $\boldsymbol{a}$ and $\boldsymbol{b}$ represent the *fibre* and *(cross-fibre) sheet* directions of the myocardium, respectively, and the *sheet-normal* direction is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. The function W enjoys the symmetry

$$W(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) = W(\lambda_2, \lambda_1, \zeta_2, \zeta_1, \xi_2, \xi_1). \tag{2.3}$$

In the reference state $\boldsymbol{U} = \boldsymbol{I}$ (the identity tensor), $\lambda_1 = \lambda_2 = \lambda_3 = 1$, any orthonormal set of vectors can represent the principal directions of $\boldsymbol{U}$. For simplicity, we let $\boldsymbol{a} = \boldsymbol{e}_3$ and $\boldsymbol{b} = \boldsymbol{e}_2$. Hence, $\zeta_1 = \zeta_2 = 0$ and $\xi_2 = 1$, $\xi_1 = 0$ in this state. To be consistent with the classical linear theory of incompressible orthotropic elasticity, appropriate for infinitesimal deformations, we must have the non-zero second derivative relations

$$\begin{aligned} &\frac{\partial^2 W}{\partial \lambda_1^2}(1, 1, 0, 0, 0, 1) = 4\mu + 4\mu_1 + \beta_1, \\ &\frac{\partial^2 W}{\partial \lambda_2^2}(1, 1, 0, 0, 0, 1) = 4\mu + 2\mu_1 + 4\mu_2 + \beta_1 + \beta_2 - 2\beta_3, \\ &\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}(1, 1, 0, 0, 0, 1) = 2\mu + 2\mu_1 + \beta_1 - \beta_3, \end{aligned} \tag{2.4}$$

where μ, μ_1, μ_2, β_1, β_2 and β_3 are ground-state elastic constants (Spencer 1984).

The classical invariants I_k, $(k = 1, 2, \ldots, 7)$ are related to our invariants via the relations

$$\begin{aligned} I_1 &= tr\boldsymbol{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{(tr\boldsymbol{C})^2 - tr\boldsymbol{C}^2}{2} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \\ I_4 &= \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a} = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3, \\ I_5 &= \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a} = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3, \\ I_6 &= \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b} = \lambda_1^2\xi_1 + \lambda_2^2\xi_2 + \lambda_3^2\xi_3, \\ I_7 &= \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b} = \lambda_1^4\xi_1 + \lambda_2^4\xi_2 + \lambda_3^4\xi_3, \end{aligned} \tag{2.5}$$

where $\boldsymbol{C} = \boldsymbol{U}^2$, $\zeta_3 = 1 - \zeta_1 - \zeta_2$ and $\xi_3 = 1 - \xi_1 - \xi_2$. For an incompressible solid, the invariant $I_3 = \det(\boldsymbol{C}) = (\lambda_1\lambda_2\lambda_3)^2 = 1$. The invariant sets $\{I_1, I_2, I_4, I_5, I_6, I_7\}$ and $\{\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2\}$ are a minimal integrity basis (Spencer 1971) with a syzygy (Shariff 2011).

3 Biaxial and simple shear deformations

In Sect. 6, we compare our theory with the biaxial and simple shear experiments of Yin et al. (1987) and Dokos et al. (2002), respectively. To facilitate Sect. 6, we give some theoretical results for biaxial and simple shear deformations and reveal the mathematical simplicity of the proposed formulation.

3.1 Homogeneous biaxial deformation

We consider the pure homogeneous deformation defined by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{3.1}$$

where x_i and X_i are the Cartesian components of $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively. For this deformation, the principal axes of the deformation coincide with the Cartesian coordinate axes and are fixed as the values of the stretches change. Thus, $\boldsymbol{F} \equiv diag(\lambda_1, \lambda_2, \lambda_3)$. On specializing to a biaxial deformation applied on a thin sheet that lies on the (X_1, X_2)-plane with the Cauchy stress component $\sigma_{33} = 0$, we have the stress–strain relations (Shariff 2011)

$$\sigma_{11} = \lambda_1 \frac{\partial W}{\partial \lambda_1}, \quad \sigma_{22} = \lambda_2 \frac{\partial W}{\partial \lambda_2}, \tag{3.2}$$

$$\begin{aligned}\sigma_{12} = \frac{2\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2} \Bigg(&\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_2} \right) \boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 \\ &+ \left(\frac{\partial W}{\partial \xi_1} - \frac{\partial W}{\partial \xi_2} \right) \boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2 \Bigg),\end{aligned} \tag{3.3}$$

$$\sigma_{\alpha 3} = \frac{2\lambda_\alpha\lambda_3}{\lambda_\alpha^2 - \lambda_3^2} \left(\frac{\partial W}{\partial \zeta_\alpha} \boldsymbol{e}_\alpha \bullet \boldsymbol{A}\boldsymbol{e}_3 + \frac{\partial W}{\partial \xi_\alpha} \boldsymbol{e}_\alpha \bullet \boldsymbol{B}\boldsymbol{e}_3 \right), \quad \alpha = 1, 2, \tag{3.4}$$

where $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$ ($\otimes$ denotes the dyadic product) and $\boldsymbol{B} = \boldsymbol{b} \otimes \boldsymbol{b}$. When the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are taken to be perpendicular to $\boldsymbol{e}_3$, we have,

$$\sigma_{\alpha 3} = 0, \quad \alpha = 1, 2. \tag{3.5}$$

In this case, it is explicit in Eq. (3.3) that σ_{12} vanishes if $\boldsymbol{a}$ or $\boldsymbol{b}$ is along one of the coordinate axes or for a mechanically equivalent material when $\zeta_\alpha = \xi_\alpha$ $(\alpha = 1, 2)$ and $\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 = -\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2$. In this case the Cauchy stress $\boldsymbol{\sigma}$ is coaxial with the left stretch tensor $\boldsymbol{V}$.

3.2 Simple shear

In Sect. 3.1, results for a homogeneous deformation, where the principal directions are fixed during deformation, are given. In this section we give results for a simple shear deformation where the principal directions of $\boldsymbol{U}$ change continuously during deformation.

To describe this deformation we let the Cartesian axes of $\boldsymbol{x}$ and $\boldsymbol{X}$ to coincide and the deformation can be described by the equations

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{3.6}$$

where *the amount of shear* $\gamma \geq 0$. The principal directions $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ have Cartesian components

$$\begin{bmatrix} c \\ s \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -s \\ c \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3.7}$$

respectively, where c and s are given in Eq. (3.9) (Shariff 2008a). It can be easily shown that the principal stretches take the values

$$\lambda_1 = \frac{\gamma + \sqrt{\gamma^2 + 4}}{2} \geq 1, \quad \lambda_2 = \frac{1}{\lambda_1} = \frac{\sqrt{\gamma^2 + 4} - \gamma}{2} \leq 1, \quad \lambda_3 = 1 \tag{3.8}$$

and

$$c = \frac{1}{\sqrt{1 + \lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1 + \lambda_1^2}}. \tag{3.9}$$

Without loss of generality, we consider $\sigma_{33} = 0$, since incompressibility allows the superposition of an arbitrary hydrostatic stress without affecting the deformation.

The Cartesian shear component of the Cauchy stress takes the form (Shariff 2011)

$$\sigma_{12} = 2\left[l_1(\gamma s^2 + cs) + l_2(\gamma c^2 - cs) + l_3 \gamma cs \right] \tag{3.10}$$

where

$$\begin{aligned} l_1 &= \frac{1}{2\lambda_1} \frac{\partial \tilde{W}}{\partial \lambda_1}, \quad l_2 = \frac{1}{2\lambda_2} \frac{\partial \tilde{W}}{\partial \lambda_2}, \\ l_3 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \Bigg[\left(\frac{\partial \tilde{W}}{\partial \zeta_1} - \frac{\partial \tilde{W}}{\partial \zeta_2} \right) \boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 \\ &\quad + \left(\frac{\partial \tilde{W}}{\partial \xi_1} - \frac{\partial \tilde{W}}{\partial \xi_2} \right) \boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2 \Bigg]. \end{aligned} \tag{3.11}$$

4 A specific form of incompressible W_e

Using series expansion and Weierstrass approximation theorem, Shariff (2011) has shown that a general strain energy function W_e for an incompressible orthotropic solid can be written in the form

$$\begin{aligned} W_e = \sum_{i=1}^{3} \hat{f}(\lambda_i, \zeta_i, \xi_i) &+ \hat{g}(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) \\ &+ \hat{g}(\lambda_1, \lambda_3, \zeta_1, \zeta_3, \xi_1, \xi_3) + \hat{g}(\lambda_2, \lambda_3, \zeta_2, \zeta_3, \xi_2, \xi_3). \end{aligned} \tag{4.1}$$

The function $\hat{g}$ has the symmetry

$$\hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j, \xi_i, \xi_j) = \hat{g}(\lambda_j, \lambda_i, \zeta_j, \zeta_i, \xi_j, \xi_i), \, i \neq j.$$

A general incompressible nonlinear (finite deformation) orthotropic strain energy function is more difficult to analyse than a (infinitesimal) linear one. For an incompressible material, the linear strain energy function has six independent ground-state constants (see Eq. (2.4)), where their role is generally fully understood. However, more often, previously proposed nonlinear strain energy functions have constants that are indirectly related to the ground-state constants, and generally, their role is not straightforward to analyse. We also note that some workers in the past proposed strain energy functions, where some of the six classical ground-state constants are not independent. In this communication, the six classical ground-state constants are assumed, on the onset, to be independent, and their specific values for a particular material are obtained experimentally. However, if there is a justification that some of the classical ground-state constants are not independent, then the proposed model could easily accommodate the interdependence of the ground-state constants (see Sect. 6).

A nonlinear strain energy function where its classical ground-state constants are explicitly expressed is attractive in the sense that their role is easier to analyse (see also Sect. 5). Using our proposed invariants, it is straightforward to extent the linear strain energy (Spencer 1984) to a semi-linear form (for mildly moderate strains), that is, the terms in W_e given by (4.1) have the forms

$$\hat{f}(\lambda_i,\zeta_i,\xi_i) = (\lambda_i-1)^2\left[\mu+2\mu_1\zeta_i+2\mu_2\xi_i + \frac{\beta_1}{2}\zeta_i^2+\frac{\beta_2}{2}\xi_i^2+\beta_3\zeta_i\xi_i\right]$$
$$\hat{g}(\lambda_i,\lambda_j,\zeta_i,\zeta_j,\xi_i,\xi_j) = (\lambda_i-1)(\lambda_j-1)[\beta_1\zeta_i\zeta_j+\beta_2\xi_i\xi_j + \beta_3(\zeta_i\xi_j+\xi_i\zeta_j)],\quad i\neq j. \tag{4.2}$$

For larger strains, we propose an extension of the semi-linear form, where

$$\hat{f}(\lambda_i,\zeta_i,\xi_i) = r(\lambda_i)\left[\mu+2\mu_1\zeta_i+2\mu_2\xi_i + \frac{\beta_1}{2}\zeta_i^2+\frac{\beta_2}{2}\xi_i^2+\beta_3\zeta_i\xi_i\right]$$
$$\hat{g}(\lambda_i,\lambda_j,\zeta_i,\zeta_j,\xi_i,\xi_j) = s(\lambda_i)s(\lambda_j)[\beta_1\zeta_i\zeta_j+\beta_2\xi_i\xi_j + \beta_3(\zeta_i\xi_j+\xi_i\zeta_j)],\quad i\neq j, \tag{4.3}$$

where $r = s^2$. It is clear from (4.3) that the strain energy function has a unique value if two or more of the principal stretches have the same value. The classical ground-state constants appear explicitly in the strain energy function. To satisfy the ground-state conditions and zero strain energy at the reference configuration, we impose the conditions $s(1) = 0$ and $s'(1) = 1$. Note that, although the semi-linear form is valid for mildly moderate strains, useful information can be extracted from it, and in view of the ground-state-constant similarity between the semi-linear and the extended forms, this information can be used to analyse the extended strain energy function. Also, since s is a single variable function, it is easier to analyse than multivariable functions. This formulation is attractive in the sense that we only need to formulate s for different types of orthotropic material, for example, fibre reinforced rubbers and orthotropic soft tissues.

In soft tissues, the initial large extension is generally achieved at relatively low levels of stress with subsequent stiffening at higher levels of extension. This behaviour is due to the recruitment of collagen fibres as they become uncrimped and reach their natural lengths (Ogden 2003; Holzapfel and Ogden 2009a). The inverse error function $erf^{-1}(x)$ seems a good candidate to describe the above mentioned soft tissue stress–strain behaviour since it has low initial gradients followed by high gradients at higher values of x. In view of this, we propose the function s given by

$$s(x) = \frac{2}{\alpha_0\sqrt{\pi}}erf^{-1}(\alpha_0\ln(x)) + \sum_{i=1}^{n}\phi_i(x), \tag{4.4}$$

where $\alpha_0 \neq 0$ is a dimensionless material parameter and the function ϕ_i has the property $\phi_i(1) = \phi_i'(1) = 0$ so that the conditions $s(1) = 0$ and $s'(1) = 1$ are satisfied. The first part of Equation (4.4) describes the primary stress–strain behaviour while the second part is used for fine-tuning. We use the natural log function in Eq. (4.4) in order to have a large range of admissible values of α_0 (see Sect. 5 for details). In this paper, to fit the experimental data considered here, we only consider

$$s(x) = \frac{2}{\alpha_0\sqrt{\pi}}erf^{-1}(\alpha_0\ln(x)) + \phi_1(x) \tag{4.5}$$

where $\phi_1(x) = \alpha_1(e^{1-x}+x-2)$ (Shariff 2000) and α_1 is a dimensionless material parameter. We note, for example, the function $\phi_1(x) = \alpha_1(x-1)^3$ can also be used to fit the experimental data reasonably; however, we do not discuss its fitting behaviour here.

5 Constraints on material constants

To ensure physically reasonable responses, restrictions are imposed on the proposed strain energy function which in turn restrict the values of the material constants. We first consider the restrictions on the ground-state constants. In incompressible infinitesimal classical elasticity, if we let $\boldsymbol{a}$ and $\boldsymbol{b}$ have the Cartesian components $[1, 0, 0]^T$ and $[0, 1, 0]^T$, respectively, we have the stress–strain relations

$$\boldsymbol{\sigma}_M = \boldsymbol{A}_m\boldsymbol{e}_M - p\boldsymbol{w}, \tag{5.1}$$

where $\boldsymbol{\sigma}_M = [\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{23},\sigma_{31},\sigma_{12}]^T$, $\boldsymbol{e}_M = [e_{11},e_{22},e_{33},2e_{32},2e_{31},2e_{12}]^T$,

$\boldsymbol{w} = [1, 1, 1, 0, 0, 0]^T$,

$$\boldsymbol{A}_M = \begin{bmatrix} c_1 & \beta_3 & 0 & 0 & 0 & 0 \\ \beta_3 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_5 \end{bmatrix}, \tag{5.2}$$

$c_1 = \beta_1 + 2\mu + 4\mu_1$, $c_2 = \beta_2 + 2\mu + 4\mu_2$, $c_3 = \mu + \mu_2$, $c_4 = \mu + \mu_1$, $c_5 = \mu + \mu_1 + \mu_2$ and e_{ij} are the Cartesian components of the infinitesimal strain.The strain energy function is given by

$$W_e = \frac{1}{2}\boldsymbol{\sigma}_M^T\boldsymbol{e}_M = \frac{1}{2}\Big[(c_1 + 2\mu)e_{11}^2 + 2(\beta_3 + 2\mu)e_{11}e_{22} + (c_2 + 2\mu)e_{22}^2 + 4c_3e_{32}^2 + 4c_4e_{31}^2 + 4c_5e_{12}^2\Big], \tag{5.3}$$

after taking into account that $e_{11} + e_{22} + e_{33} = 0$. Since $e_{11}, e_{22}, e_{12}, e_{31}$ and e_{32} are independent, necessary and sufficient conditions for (5.3) to be positive definite are:

$$c_3 > 0, c_4 > 0, c_5 > 0, c_1 + 2\mu > 0, \quad (c_1 + 2\mu)(c_2 + 2\mu) > (\beta_3 + 2\mu)^2. \tag{5.4}$$

Sufficient conditions for W_e to be positive definite in a finite deformation are given in "Appendix A".

The restrictions on the values of the parameters α_0 and α_1 are governed by the restrictions on the function r or s. We do this by considering a special set of admissible ground-state constant values, where $\mu > 0$ and the rest have zero values. This set of values corresponds to a strain energy of an isotropic material. Using Hill (1970) inequality, it is shown in Shariff (2000) that, to ensure physically reasonable responses for incompressible isotropic materials, we require the condition $h'(x) > 0$, for $x > 0$, where $h(x) = xr'(x)$; in this paper, we use this *necessary* condition to restrict the values of α_0 and α_1 for the proposed anisotropic model. For example, if we let

$$s(x) = \frac{2}{\alpha_0\sqrt{\pi}} erf^{-1}(\alpha_0 \ln(x)), \tag{5.5}$$

we then have, for $x > 0$,

$$h'(x) = 2xs'(x)^2 + \frac{4}{x}z^2e^{2z^2} > 0, \tag{5.6}$$

for all $\alpha_0 \neq 0$, where $z = erf^{-1}(\alpha_0 \ln(x))$. In the case of $s(x)$ given in (4.5), the admissible ranges of α_0 and α_1 are not straightforward to obtain. However, for given values of α_0 and α_1, we can easily (and non-rigorously) verify if $h'(x) > 0$ by plotting $h'(x)$ for practical values of $x > 0$. The concepts of polyconvexity (Itskov and Aksel 2004), convexity and stability (Holzapfel and Ogden 2009a) can also be used to restrict the values of our material constants, and we hope to do this in the near future. However, we note that stability in an infinitesimal deformation (relative to a stress-free ground-state configuration) is achieved if the classical ground-state constants have the restricted values.

6 Comparison with experimental data

In this section, we show the efficacy of the special constitutive form for fitting data on the myocardium. The simple shear data of Dokos et al. (2002) and the biaxial data of Yin et al. (1987) are used in the curve fitting exercise. The least-squares method is used in the curve fitting, where its solution corresponds to a local minimum. We cannot guarantee that our local minimum is a global minimum. We emphasize that care must be taken in interpreting the results from a curve fitting exercise. For example, the ground-state constant values will not be accurately obtained if there are insufficient data at low strains. In this communication, the experimental data for both the biaxial and simple shear deformations are extracted from the published work of Holzapfel and Ogden (2009a).

6.1 Dokos et al. (2002) experiment

The theoretical curves used to fit the experiment data are obtained from Eqs. (3.10) and (3.11). We model the experimental shear in different directions with a shear in one direction and the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ acquiring different directions.

In Fig. 1, there are six sets of data; however, the data corresponding to the fibre/sheet directions with Cartesian components $[1, 0, 0]^T/[0, 0, 1]^T$ and $[0, 0, 1]^T/[1, 0, 0]^T$ are indistinguishable. We note that no experiment is perfect. This indistinguishable behaviour could be caused by minute errors or approximations in the experiment, or it could be the actual behaviour of the myocardium specimen or other unknown factors. The model we proposed here is intended to characterize several types of myocardium, not only the specimen used in Dokos et al. (2002) experiment. Hence, we do not, on the onset, construct a constitutive equation so that the two theoretical curves, corresponding to the fibre/sheet components $[1, 0, 0]^T/[0, 0, 1]^T$ and $[0, 0, 1]^T/[1, 0, 0]^T$, are the same. We note that, Holzapfel and Ogden (2009a), however, remove the invariant $\boldsymbol{a} \bullet \boldsymbol{C}(\boldsymbol{a} \times \boldsymbol{b})$ (or $\boldsymbol{b} \bullet \boldsymbol{C}(\boldsymbol{a} \times \boldsymbol{b})$) from their strain energy function so that their two theoretical curves will be the same. In infinitesimal deformations, the shear stresses for the fibre/sheet directions $[1, 0, 0]^T/[0, 0, 1]^T$ and $[0, 0, 1]^T/[1, 0, 0]^T$ are

$$\sigma_{12} = (\mu + \mu_1)\gamma \tag{6.1}$$

and

$$\sigma_{12} = (\mu + \mu_2)\gamma, \tag{6.2}$$

respectively. They are the same if and only if $\mu_1 = \mu_2$.

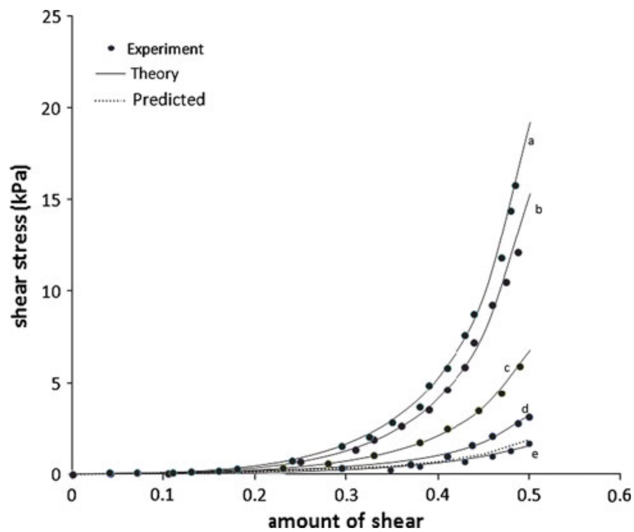

Fig. 1 Fit of the proposed model to the loading data of Dokos et al. (2002) for simple shear in various directions. $\alpha_0 = 3.416$, $\alpha_1 = 0.437$, $\mu = 0.353$, $\mu_1 = -0.267$, $\mu_2 = 0.109$, $\beta_1 = 95.224$, $\beta_2 = 8.773$, $\beta_3 = -25$. Eight material parameters

We only apply the least-squares fit to five sets of data that correspond to fibre/sheet directions with Cartesian components (from top to bottom in Fig. 1): (a) $[0, 1, 0]^T/[1, 0, 0]^T$ (b)$[0, 1, 0]^T/[0, 0, 1]^T$ (c) $[1, 0, 0]^T/[0, 1, 0]^T$ (d) $[0, 0, 1]^T/[0, 1, 0]^T$(e)$[1, 0, 0]^T/[0, 0, 1]^T$. We then predict the set of data that corresponds to the fibre/sheet directions with components $[0, 0, 1]^T/[1, 0, 0]^T$. Initially, we blindly least-square fit all the eight material parameters to the five sets of data and obtained ground-state values that do not satisfy the last inequality in (5.4); we do not expect to get reasonable ground-state results from blind fitting since some of the data at low strains are missing or indistinguishable for the different fibre/sheet directions. Since these are not desirable values, we fix the value $\beta_3 = -25$ in an ad hoc manner and obtain the rest of the material parameter values via the least-squares method. These eight material parameter values are given in Fig. 1, and with these values, the inequalities in (5.4) are satisfied and the condition $h'(x) > 0$ is satisfied. It is clear in Fig. 1 that very good agreement is indicated between the model and the experimental data. The predicted curve is also in good agreement with the experimental data. It is worth noting that a reasonable agreement between theory and experiment can also be obtained if we use seven parameters, where the function ϕ_1 is omitted in Eq. (4.5). However, we do not show this plot.

It is desirable, in general, to curve fit a data using a least-squares method with a small number of parameters. One way to reduce the number of material parameters is by assuming that some of them are not independent. In the past, some constitutive models have ground-state constants that are numerically less than six which indicate that, in these models, either some of the classical ground-state constants are assumed to be zero or some of them are not independent. For example, in "Appendix B", we have shown that, in Holzapfel and Ogden (2009a) model, only four of the classical ground-state constants are independent, that is,

$$\mu = a_{iso} - a_{fs}, \mu_1 = a_{fs}, \mu_2 = a_{fs},$$
$$\beta_1 = 4a_f, \beta_2 = 4a_s, \beta_3 = 2a_{fs}, \tag{6.3}$$

where a_{iso}, a_{fs}, a_f and a_s are independent parameters. If we substitute Eq. (6.3) in the proposed model, we then have a model with only six independent parameters. It is clear in Fig. 2 that the reduced six parameter model fits Dokos et al. (2002) data very well. The six-parameter *predicted* curve fits better then the eight-parameter *predicted* curve. Since our model is different from Holzapfel and Ogden (2009a) model, this good fit seems to further justify Holzapfel and Ogden (2009a) analysis of passive myocardium.

6.2 Yin et al. (1987) experiment

Holzapfel and Ogden (2009a) used the biaxial data of Yin et al. (1987) for illustration purposes because, to their knowledge, they are the only true biaxial data available; we will do the same. We note that Yin et al. (1987) data do not provide information at low strains, hence will not give accurate results for the classical ground-state constants. For the biaxial test, the relevant stress components take the forms:

$$S_{ff} = A_1\frac{r'(\lambda_1)}{\lambda_1} - \mu\frac{\lambda_3 r'(\lambda_3)}{\lambda_1^2} + \beta_3\frac{s'(\lambda_1)s(\lambda_2)}{\lambda_1^2}$$
$$S_{ss} = A_2\frac{r'(\lambda_2)}{\lambda_2} - \mu\frac{\lambda_3 r'(\lambda_3)}{\lambda_2^2} + \beta_3\frac{s'(\lambda_2)s(\lambda_1)}{\lambda_2^2}, \tag{6.4}$$

where

$$A_1 = \mu + 2\mu_1 + \frac{\beta_1}{2}, A_2 = \mu + 2\mu_2 + \frac{\beta_2}{2}, \tag{6.5}$$

$\lambda_1 = \sqrt{2E_{ff}+1}$, $\lambda_2 = \sqrt{2E_{ss}+1}$, S_{ss} and S_{ff} are the components of the second Piola-Kirchhoff stress in the sheet (cross-fibre) and fibre directions, respectively, and E_{SS} and E_{ff} are the corresponding components of the Green-Lagrange strain tensor. Curve fitting will only give numerical values of A_1,A_2, μ, β_3, α_0 and α_1. It is clear from (6.5) that we cannot fully characterize the strain energy function from the biaxial data in the sheet (cross-fibre) and fibre directions (see also Shariff 2008a and Holzapfel and Ogden 2009b). Hence, care must be taken in drawing conclusions from this biaxial data. This highlights the need for more complete biaxial data, for example, in addition to biaxial data in the sheet and fibre directions, we need stress–strain biaxial data sets that are not in the sheet or fibre directions (Shariff 2008a) (the author is not sure if this can be done experimentally). Due to lack of data at low strains, the fit presented in Figs. 3

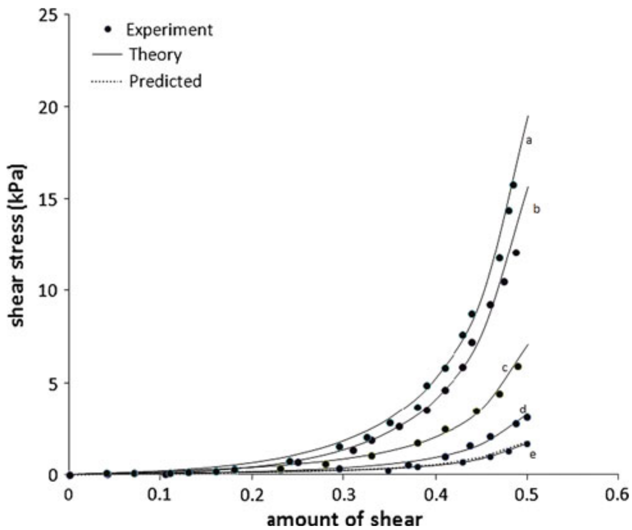

Fig. 2 Fit of the proposed model to the loading data of Dokos et al. (2002) for simple shear in various directions. $\alpha_0 = 3.617$, $\alpha_1 = 2.166$, $a_{iso} = 0.385$, $a_{fs} = 0.749$, $a_s = 0.335$, $a_f = 9.067$. Six material parameters

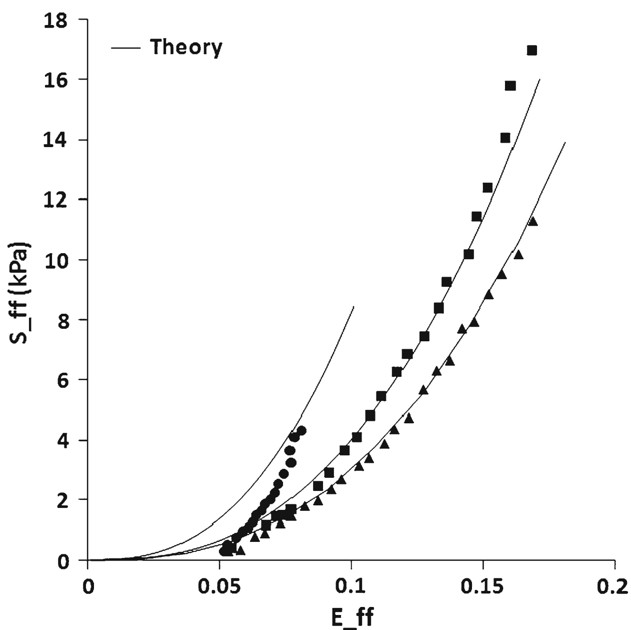

Fig. 3 Fit of the proposed model to the biaxial data of Yin et al. (1987). S_ff $= S_{ff}$, E_ff $= E_{ff}$ and $\frac{E_{ff}}{E_{ss}} = 2.05$ (*triangles*), 1.02 (*squares*) and 0.48 (*circles*). $\alpha_0 = 4.998$, $\alpha_1 = 250.819$, $A_1 = 0.047$, $\mu = 0$, $\beta_3 = 0.061$

and 4 is therefore rather crude; however, our model reflects the general behaviour for the different constant strain ratios $\frac{E_{ff}}{E_{ss}}$.

The biaxial data of Yin et al. (1987) can also be captured (plot not shown) by a transversely isotropic model with the preferred direction $\boldsymbol{a}$ using only three parameters A_1, μ and

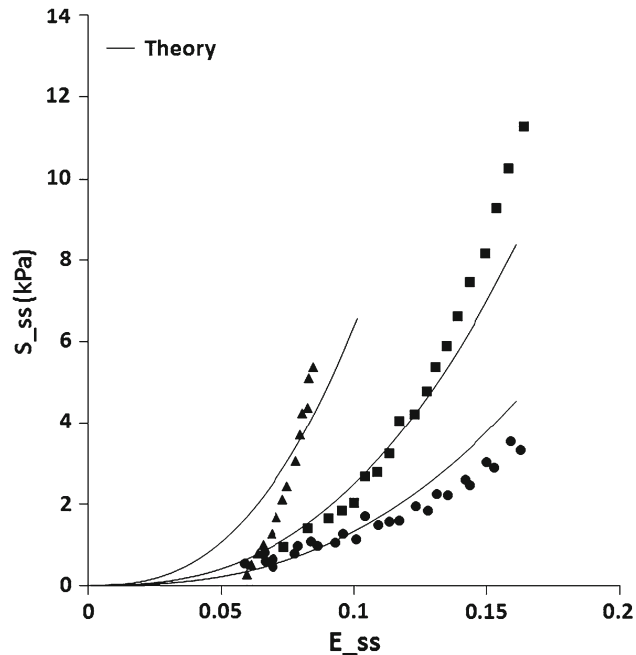

Fig. 4 Fit of the proposed model to the biaxial data of Yin et al. (1987). S_ss $= S_{ss}$, E_ss $= E_{ss}$ and $\frac{E_{ff}}{E_{ss}} = 2.05$ (*triangles*), 1.02 (*squares*) and 0.48 (*circles*). $\alpha_0 = 4.998$, $\alpha_1 = 250.819$, $A_2 = 0.017$, $\mu = 0$, $\beta_3 = 0.061$

α_0. The stress–strain component relations for a transversely isotropic solid take the form

$$S_{ff} = A_1 \frac{r'(\lambda_1)}{\lambda_1} - \mu \frac{\lambda_3 r'(\lambda_3)}{\lambda_1^2} \tag{6.6}$$

$$S_{ss} = \mu \left(\frac{r'(\lambda_2)}{\lambda_2} - \frac{\lambda_3 r'(\lambda_3)}{\lambda_2^2} \right). \tag{6.7}$$

Curve fitting will only give numerical values for A_1, μ and α_0. Hence the values of μ_1 and β_1 cannot be uniquely obtained. This suggest that, even for transversely isotropic materials [Yin et al. (1987) material is orthotropic], we need additional sets of experimental data that are not parallel or perpendicular to $\boldsymbol{a}$ to fully characterize the material.

7 Concluding remarks

A physical invariant general strain energy function containing all the orthotropic classical ground-state constants is proposed. The general form contains only a general single variable function which is then specialized to characterize passive myocardium. The invariants used in our constitutive equation have immediate physical interpretation which can facilitate experiments to obtain specific forms of the strain energy. The single general function may be easily specialized to mechanically describe other orthotropic soft tissues and can be easily adapted to model stress-softening behaviour of soft tissues with anisotropic behaviour (Shariff 2008b). The proposed constitutive model fitted well with experimental data and managed to predict a set of simple shear experimental data. The extent of the proposed model applicability to other orthotropic soft tissues needs to be assessed by comparing it with relevant experimental data of a much wider class of orthotropic soft tissues.

Appendix A

In this appendix, we derive, for finite strain deformations, sufficient conditions for the strain energy W_e to be positive definite. Consider the tensor $\boldsymbol{S}$ (coaxial with the right stretch tensor $\boldsymbol{U}$) with the eigenvalues $s(\lambda_i)$, that is,

$$\boldsymbol{S} = \sum_{i=1}^{3} s(\lambda_i)\boldsymbol{e}_i \otimes \boldsymbol{e}_i. \tag{A.1}$$

Hence, we have

$$\boldsymbol{S}^2 = \sum_{i=1}^{3} r(\lambda_i)\boldsymbol{e}_i \otimes \boldsymbol{e}_i.$$

With a little algebra, we have,

$$W_e = \mu tr(\boldsymbol{S}^2) + 2\mu_1 \boldsymbol{a} \bullet \boldsymbol{S}^2\boldsymbol{a} + 2\mu_2 \boldsymbol{b} \bullet \boldsymbol{S}^2\boldsymbol{b} + \frac{\beta_1}{2}(\boldsymbol{a} \bullet \boldsymbol{S}\boldsymbol{a})^2 + \frac{\beta_2}{2}(\boldsymbol{b} \bullet \boldsymbol{S}\boldsymbol{b})^2 + \beta_3(\boldsymbol{a} \bullet \boldsymbol{S}\boldsymbol{a})(\boldsymbol{b} \bullet \boldsymbol{S}\boldsymbol{b}). \tag{A.2}$$

if we let $\boldsymbol{a}$ and $\boldsymbol{b}$ have the Cartesian components $[1, 0, 0]^T$ and $[0, 1, 0]^T$, respectively, we then have

$$W_e = \frac{1}{2}\boldsymbol{s}_m^T \boldsymbol{A}_m \boldsymbol{s}_m, \tag{A.3}$$

where $\boldsymbol{s}_m = [S_{11}, S_{22}, S_{33}, 2S_{32}, 2S_{31}, 2S_{12}]^T$ and S_{ij} are the Cartesian components of $\boldsymbol{S}$. We note that, due to the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$, some of the S_{ij} are not independent, and in view of this, necessary and sufficient conditions for W_e to be positive definite are not trivial to obtain. However, if all the eigenvalues of the matrix $\boldsymbol{A}_m$ are positive, then W_e in (A.3) is positive definite. Hence, sufficient conditions for positive definite W_e are:

$$c_1 > 0, \quad c_1c_2 > \beta_3^2, \quad \mu > 0, \quad c_3 > 0, \quad c_4 > 0, \quad c_5 > 0. \tag{A.4}$$

Appendix B

The strain energy function proposed by Holzapfel and Ogden (2009a) is

$$W_e = \frac{a_{iso}}{b_{iso}} \exp\left[b_{iso}(I_1 - 3)\right] + \sum_{i=f,s} \frac{a_i}{2b_i}\left(\exp\left[b_i(I_{4i} - 1)^2\right] - 1\right) + \frac{a_{fs}}{b_{fs}}\left(\exp\left[b_{fs}I_{8fs}^2\right] - 1\right), \tag{B.1}$$

where $I_{4f} = I_4$, $I_{4s} = I_6$ and

$$I_{8fs} = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{b} = \sum_{i=1}^{3} \lambda_i^2(\boldsymbol{a} \bullet \boldsymbol{e}_i)(\boldsymbol{b} \bullet \boldsymbol{e}_i). \tag{B.2}$$

In view of (2.5) and (B.2), and the incompressibility constraints $\lambda_1\lambda_2\lambda_3 = 1$, we can write (B.1) as

$$W_e = W_H(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2). \tag{B.3}$$

In order to extract the ground-state constants for (B.1), we consider the second derivative of (B.3) at the reference state, that is,

$$\frac{\partial^2 W_H}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_2, \xi_1, \xi_2) = 4a_{iso} - 4a_{fs} + 4a_{fs}(\zeta_1 + \zeta_3) + 4a_{fs}(\xi_1 + \xi_3) + 4a_f(\zeta_1 - \zeta_3)^2 + 4a_s(\xi_1 - \xi_3)^2 + 4a_{fs}(\zeta_1 - \zeta_3)(\xi_1 - \xi_3). \tag{B.4}$$

The above equation is obtained using the relations

$$\boldsymbol{a} \bullet \boldsymbol{b} = (\boldsymbol{a} \bullet \boldsymbol{e}_1)(\boldsymbol{b} \bullet \boldsymbol{e}_1) + (\boldsymbol{a} \bullet \boldsymbol{e}_2)(\boldsymbol{b} \bullet \boldsymbol{e}_2) + (\boldsymbol{a} \bullet \boldsymbol{e}_3)(\boldsymbol{b} \bullet \boldsymbol{e}_3) = 0 \tag{B.5}$$

and

$$2(\boldsymbol{a} \bullet \boldsymbol{e}_1)(\boldsymbol{b} \bullet \boldsymbol{e}_1)(\boldsymbol{a} \bullet \boldsymbol{e}_3)(\boldsymbol{b} \bullet \boldsymbol{e}_3) = \zeta_2\xi_2 - \zeta_1\xi_1 - \zeta_3\xi_3. \tag{B.6}$$

However, for a general orthotropic incompressible strain energy function $W(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2)$, we have

$$\frac{\partial^2 W}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_2, \xi_1, \xi_2) = 4\mu + 4\mu_1(\zeta_1 + \zeta_3) + 4\mu_2(\xi_1 + \xi_3) + \beta_1(\zeta_1 - \zeta_3)^2 + \beta_2(\xi_1 - \xi_3)^2 + 2\beta_3(\zeta_1 - \zeta_3)(\xi_1 - \xi_3). \tag{B.7}$$

Since in the reference state ζ_1, ζ_2, ξ_1 and ξ_2 are arbitrary, on comparing (B.4) and (B.7), we have

$$\mu = a_{iso} - a_{fs}, \mu_1 = a_{fs}, \mu_2 = a_{fs}, \beta_1 = 4a_f, \beta_2 = 4a_s, \beta_3 = 2a_{fs}. \tag{B.8}$$

Hence, in Holzapfel and Ogden (2009a) model, only four of the six classical ground-state constants are independent.

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